

ON THE EFFECTIVE INERTIAL MASS DENSITY OF A DISSIPATIVE FLUID

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Abstract

It is shown that the effective inertial mass density of a dissipative fluid just after leaving the equilibrium, on a time scale of the order of relaxation time, reduces by a factor which depends on dissipative variables. Prospective applications of this result to cosmological and astrophysical scenarios are discussed.

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1 Introduction

In classical dynamics the inertial mass is defined as the factor of proportionality between the three-force applied to a particle (a fluid element) and the resulting three-acceleration, according to Newton's second law.

In relativistic dynamics a similar relation only holds (in general) in the instantaneous rest frame (i.r.f.), since the three-acceleration and the force that causes it are not (in general) parallel, except in the i.r.f. (see for example [1]).

In this work we shall derive an expression for the effective inertial mass density of a dissipative relativistic fluid, which is valid just after the fluid leaves the equilibrium, on a time scale of the order of relaxation time.

By "effective inertial mass" (e.i.m.) density we mean the factor of proportionality between the applied three-force density and the resulting proper acceleration (i.e., the three-acceleration measured in the i.r.f.).

As we shall see, the obtained expression for the e.i.m. density contains a contribution from dissipative variables which reduces its value with respect to the non-dissipative situation. Such decreasing of e.i.m. density has already been brought out in the spherically symmetric self-gravitating case [2], the axially symmetric self-gravitating case [3] and also for slowly rotating self-gravitating systems [4]. Here we want to provide a general derivation for the e.i.m. density of a dissipative fluid, which is independent on the symmetry of the problem. On the other hand, in order to derive the expression for the e.i.m. density, it will be necessary to suppress gravitational contributions to four acceleration. This will be achieved by evaluating the equation of motion in a locally Minkowski frame, where local gravitational effects vanish. Then evaluating the resulting equation of motion in the i.r.f. we are led to an expression which takes the desired, "Newtonian", form.

$$\text{Force} = \text{e.i.m.} \times \text{acceleration(proper)},$$

It is perhaps worth noticing that the concept of effective inertial mass is familiar in other branches of physics, thus for example the e.i.m. of an electron moving under a given force through a crystal, differs from the value corresponding to an electron moving under the same force in free space, and may even become negative (see [5, 6]).

In order to illustrate our point and to establish the notation, we shall first consider, very briefly, the perfect fluid case. However before closing this

Section, the following comments are in order:

It is worth noticing, that there is not a unique way to define the four velocity of a dissipative fluid. Indeed, we may take, among other possibilities, the four velocity vector as being related to the velocity of energy transport (Landau-Lifshitz [7]) or to the velocity of particles (fluid element) as done by Eckart ([8]), both choices are of course perfectly equivalent, and both lead to different definitions of rest frame. In the former case there is not energy flux in the rest frame (the T^{0j} components of the energy tensor vanish), whereas in the later the spatial components of the particle current four vector vanish in the comoving frame. In this work we adopt the Eckart choice.

This is motivated by the fact that the very concept of e.i.m. as defined above, requires (and makes sense only for) such choice (Eckart). Indeed, observe that the concept of e.i.m. requires the existence of a frame where the three-force applied to a particle and the resulting three acceleration are parallel, and this only happens in the i.r.f. where a given fluid element is at rest. But this is the Eckart frame by definition.

In the Landau-Lifshitz frame we cannot say that such relation between the applied three-force and the resulting three accelerations follows, and therefore in such case the very meaning of e.i.m. is uncertain.

2 The e.i.m. density of a relativistic perfect fluid

Let us consider a relativistic perfect fluid, whose energy-momentum tensor takes the usual form

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu} \quad (1)$$

(we use relativistic units and signature -2)

Then the equation of motion yields

$$(\rho + p)a^\alpha = h^{\alpha\nu}p_{,\nu} \quad (2)$$

with

$$h_\mu^\alpha \equiv \delta_\mu^\alpha - u^\alpha u_\mu \quad (3)$$

$$\dot{u}^\mu \equiv a^\mu = u^\nu u^\mu_{;\nu} \quad (4)$$

where colon and semicolon denote partial and covariant derivatives, respectively.

Observe that for the sake of generality we are including gravitational field, although, as already noticed, the final expression for e.i.m. density will be independent of gravity.

Eq.(2) suggests that $(\rho+p)$ plays the role of the e.i.m. density of the fluid (which is in fact the case). However, we recall that the applied three-force and the resulting three-acceleration are parallel only in the i.r.f., therefore, in order to conclude that $(\rho+p)$ represents the e.i.m. density, (2) has to be evaluated in such a frame. Furthermore, it should be observed that a^α contains both, gravitational (in the self-gravitating case) and “kinematical” contributions. Since in the concept of e.i.m. only the later should appear, we have to evaluate (2) not only in the i.r.f., but also in the locally Minkowskian frame (l.m.f.) where local gravitational effects vanish. Formally, this may be achieved, either by introducing locally Minkowskian coordinates or equivalently, by considering a tetrad field attached to such l.m.f.. Denoting by a tilde the components of tensors in the l.m.f., equation (2) reads

$$(\rho+p)\tilde{a}^\mu = \tilde{h}^{\mu\nu}p_{,\nu} \quad (5)$$

Evaluating (5) in the i.r.f., it becomes clear that $(\rho+p)$ defines the e.i.m. density of any fluid element (a well known fact, see for example [9]).

In order to better illustrate this point, let us consider the spherically symmetric case.

Thus, let the line element be

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (6)$$

with $\nu = \nu(r, t)$ and $\lambda = \lambda(r, t)$, then locally Minkowski coordinates may be introduced by [10]

$$dT = e^{\nu/2}dt \quad dx = e^{\lambda/2}dr \quad dy = rd\theta \quad dz = r\sin\theta d\phi \quad (7)$$

satisfying

$$ds^2 = dT^2 - dx^2 - dy^2 - dz^2 \quad (8)$$

Next, the four-velocity vector for any fluid element, in the frame of (6) is

$$u^\mu = \left(\frac{e^{-\nu/2}}{\sqrt{1-\omega^2}}, \frac{\omega e^{-\lambda/2}}{\sqrt{1-\omega^2}}, 0, 0 \right) \quad (9)$$

where ω is the radial velocity of the fluid element as measured in the l.m.f.

Therefore, in the l.m.f. the components of (9) are

$$\tilde{u}^\mu = \frac{\delta_T^\mu}{\sqrt{1 - \omega^2}} + \frac{\omega \delta_x^\mu}{\sqrt{1 - \omega^2}} \quad (10)$$

and the components of the four-acceleration in the l.m.f. read

$$\tilde{a}_\mu = \tilde{u}^\nu \tilde{u}_{\mu;\nu} \quad (11)$$

Then, feeding back (11) into (5), and evaluating in the i.r.f. ($\omega = 0$), we obtain the “Newtonian-type” equation

$$(\rho + p)\omega_{,T} = -p_{,x} \quad (12)$$

indicating that $(\rho + p)$ represents the e.i.m. density of the fluid.

Let us now turn to the dissipative case.

3 The e.i.m. density of a dissipative fluid

For simplicity we shall consider the case of pure heat conduction, neglecting viscosity terms. Then, the energy-momentum tensor takes the usual form

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu \quad (13)$$

and the equation of motion, projected onto the plane orthogonal to u^μ reads

$$(\rho + p)a^\alpha - p_{,\nu}h^{\nu\alpha} + h_\mu^\alpha q^\mu_{;\nu}u^\nu + q^\alpha u^\nu_{;\nu} + q^\nu u^\mu_{;\nu}h_\mu^\alpha = 0 \quad (14)$$

where q^μ , satisfying $q^\mu u_\mu = 0$, denotes the heat flow vector.

We have now to adopt a transport equation for q^μ . Although this is still a matter of discussion, it should be borne in mind that some kind of hyperbolic dissipative theory [11] should be applied in order to avoid violation of causality and other undesirable consequences derived from the (parabolic) Eckart-Landau approach. Furthermore, since we are interested in the transient regime between two equilibrium states we are compelled to use a hyperbolic theory, for it is known that the parabolic theory assumes that the system is relaxed at all times.

Thus, we shall consider the Israel-Stewart transport equation [11] (although any hyperbolic equation yielding a Cattaneo-type [11] equation in the non-relativistic limit would lead to the same result)

$$\tau h_\nu^\mu \dot{q}^\nu + q^\mu = \kappa h^{\mu\nu} (T_{,\nu} - T a_\nu) - \frac{1}{2} \kappa T^2 \left(\frac{\tau u^\alpha}{\kappa T^2} \right)_{;\alpha} q^\mu + \tau \omega^{\mu\nu} q_\nu \quad (15)$$

where τ, κ, T and $\omega^{\mu\nu}$ denote the relaxation time, the thermal conductivity, the temperature and the vorticity tensor, respectively.

Then, feeding back (15) into (14) one obtains

$$\begin{aligned} & a^\alpha \left(\rho + p - \frac{\kappa T}{\tau} \right) - p_{,\nu} h^{\nu\alpha} - \frac{1}{\tau} (q^\alpha - \kappa h^{\alpha\nu} T_{,\nu}) \\ & - \frac{1}{2\tau} \kappa T^2 \left(\frac{\tau u^\beta}{\kappa T^2} \right)_{;\beta} q^\alpha + 2\omega^{\alpha\nu} q_\nu + q^\alpha \Theta + q^\nu \sigma_\nu^\alpha = 0 \end{aligned} \quad (16)$$

with

$$\Theta = u_{;\nu}^\nu$$

$\sigma_{\mu\nu}$ as usual denotes the shear tensor, and where the relation [12]

$$u_{\alpha;\beta} = \omega_{\alpha\beta} + \sigma_{\alpha\beta} - \frac{1}{3} \Theta h_{\alpha\beta} - a_\alpha u_\beta \quad (17)$$

has been used.

We shall now consider that our system is initially, either in hydrostatic equilibrium without dissipation, or very close to equilibrium with “small” dissipation (quasi-stationary regime). Next, let us suppose that at some moment (say $t = \tilde{t}$) it leaves the initial state entering into a dissipative regime.

Then, “immediately” after departure from equilibrium (or quasi-equilibrium), where “immediately” means on a time scale of the order of τ (or smaller), the last four terms in (16) can be neglected, and we have

$$a^\alpha \left(\rho + p - \frac{\kappa T}{\tau} \right) - p_{,\nu} h^{\nu\alpha} - \frac{1}{\tau} (q^\alpha - \kappa h^{\alpha\nu} T_{,\nu}) = 0 \quad (18)$$

The components of (18) in the l.m.f. are

$$\tilde{a}^\mu \left(\rho + p - \frac{\kappa T}{\tau} \right) - p_{,\nu} \tilde{h}^{\mu\nu} - \frac{1}{\tau} (\tilde{q}^\mu - \kappa \tilde{h}^{\mu\nu} T_{,\nu}) = 0 \quad (19)$$

However, in the l.m.f., the components of the heat flux vector in the i.r.f., in the quasi-stationary regime, satisfy the “Maxwell-Fourier” equation

$$\tilde{q}^\mu = \kappa \tilde{h}^{\mu\nu} T_\nu \quad (20)$$

Indeed, just after leaving the equilibrium, the absolute value of the heat flow vector is the same as that corresponding to the quasistationary regime, and furthermore in that regime (in the l.m.f. and in the i.r.f.) both the kinematical and gravitational contributions to the four acceleration vanish, thereby justifying (20). Therefore, just after leaving the quasi-stationary regime, the last term in (19) vanishes, and this equation becomes

$$\tilde{a}^\mu \left(\rho + p - \frac{\kappa T}{\tau} \right) = p_{,\nu} \tilde{h}^{\mu\nu} \quad (21)$$

implying that the e.i.m. density is given by

$$\rho + p - \frac{\kappa T}{\tau} \quad (22)$$

or,

$$(\rho + p)(1 - \alpha) \quad (23)$$

with

$$\alpha \equiv \frac{\kappa T}{\tau(\rho + p)} \quad (24)$$

4 Discussion

We have seen that just after leaving the equilibrium the e.i.m. density of the dissipative fluid reduces its value (as compared to the non-dissipative situation) by the factor

$$1 - \alpha \quad (25)$$

giving rise, in principle, to the possibility of vanishing ($\alpha = 1$) or even negative ($\alpha > 1$) e.i.m. density. As mentioned in the Introduction, such effect was already found in some specific examples, here it appears as a general result for any dissipative fluid (self-gravitating or not).

It should be noticed that since the system is evaluated immediately (in the sense explained above) after leaving the equilibrium (or quasi-equilibrium)

then the physical meaning of thermodynamical variables (their values), are extrapolated from the state of equilibrium, and therefore any discussion about difficulties of interpreting such variables in a transient regime, is out of the point.

Also, it should be observed that causality and stability conditions hindering the system to attain condition $\alpha = 1$, are obtained on the basis of a linear approximation, whose validity, close to the critical point ($\alpha = 1$), is questionable [13].

At any rate, examples of fluids attaining the critical point and exhibiting reasonable physical properties have been presented elsewhere [14, 15].

In order to evaluate α , let us turn back to c.g.s. units. Then, assuming for simplicity $\rho + p \approx 2\rho$, we obtain

$$\frac{\kappa T}{\tau(\rho + p)} \approx \frac{[\kappa][T]}{[\tau][\rho]} \times 10^{-42} \quad (26)$$

where $[\kappa]$, $[T]$, $[\tau]$, $[\rho]$ denote the numerical values of these quantities in $\text{erg} \cdot \text{s}^{-1} \text{cm}^{-1} \text{K}^{-1}$, K , s and $\text{g} \cdot \text{cm}^{-3}$, respectively.

Obviously, this will be a very small quantity (compared to 1), unless conditions for extremely high values of κ and T are attained. In this respect, observe that although small values of τ increases (26), they are of little help since (22) is valid only on a time scale of the order of τ . Therefore, a decreasing of e.i.m. density with physically relevant consequences requires values of (26) close to the unit, due to large values of κ and T but non-negligible values of τ .

At present we may speculate that α may decrease substantially (for non-negligible values of τ) in a pre-supernovae event

Indeed, at the last stages of massive star evolution, the decreasing of the opacity of the fluid, from very high values preventing the propagation of photons and neutrinos (trapping [16]), to smaller values, gives rise to radiative heat conduction. Under these conditions both κ and T could be sufficiently large as to imply a substantial increase of α . Indeed, the values suggested in [17] ($[\kappa] \approx 10^{37}$; $[T] \approx 10^{13}$; $[\tau] \approx 10^{-4}$; $[\rho] \approx 10^{12}$) lead to $\alpha \approx 1$. The obvious consequence of which would be to enhance the efficiency of whatever expansion mechanism, of the central core, at place, because of the decreasing of its e.i.m. density. At this point it is worth noticing that the relevance of relaxational effects on gravitational collapse has been recently exhibited and stressed (see [18], and references therein)

It is also worth noticing that the inflationary equation of state (in the perfect fluid case) $\rho + p = 0$, is, as far as the equation of motion is concerned, equivalent to $\alpha = 1$ in the dissipative case (both imply the vanishing of the e.i.m. density).

Also observe, that if we impose the equation of state $\rho + p = 0$, in the dissipative case, then the resulting e.i.m. density is negative ($-\frac{\kappa T}{\tau}$), implying that the attractive gravitational force leads to an overall expansion.

Finally, it is worth stressing that it is the first term on the left in (15), the direct responsible for the decreasing in the e.i.m density, all terms including kinematical variables vanishing within the time scale considered (after leaving equilibrium). Therefore any hyperbolic dissipative theory yielding a Cattaneo-type equation in the non-relativistic limit, is expected to give a result similar to the one obtained here.

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